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Ensemble teleportation under suboptimal conditions

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Abstract
The possibility of teleportation is certainly the most interesting consequence of quantum non-separability. In the present paper, the feasibility of teleportation is examined on the basis of the rigorous ensemble interpretation of quantum mechanics if *non-ideal* constraints are imposed on the teleportation scheme. Importance is attached both to the case of noisy Einstein–Podolsky–Rosen (EPR) ensembles and to the conditions under which automatic teleportation is still possible. The success of teleportation is discussed using a new fidelity measure which avoids the weaknesses of previous proposals.

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1. Introduction

The possibility of teleportation is surely the most interesting consequence of quantum non-separability. Bennett *et al* were the first who had realized that an "unknown quantum state $|\phi\rangle$ can be disassembled into, then later reconstructed from, purely classical information and purely nonclassical Einstein–Podolsky–Rosen (EPR) correlations. To do so the sender, 'Alice', and the receiver, 'Bob', must prearrange the sharing of an EPR-correlated pair of particles. Alice makes a joint measurement on her EPR particle and the unknown quantum system, and sends Bob the classical result ... Knowing this, Bob can convert the state of his EPR particle into an exact replica of the unknown state... which Alice destroyed" [1]. Meanwhile, several experiments have been performed successfully by different research groups, thereby demonstrating the feasibility of teleportation [2–7]. With the exception of [7], all experiments employ photons both for the generation of the EPR pair and to materialize the unknown state. There are a couple of proposals to realize teleportation using massive entities, i.e., atoms in high-Q cavities [8–11], solid-state systems [12], and clouds of atoms [13, 14]. First experimental tests using pairs of calcium [15] and beryllium ions [16], respectively, have been performed successfully. A crucial point of every experimental implementation of a theoretical teleportation scheme is the joint measurement of Alice's half of the EPR pair and the unknown entity the state of which shall be teleported [17]. At present, new ideas are being discussed to overcome these difficulties [18–20], and numerous researchers are investigating the teleportation fidelity if non-ideal EPR pairs are used (see e.g. [21–28]).

In previous papers, one of us (TK) has shown that the so-called *rigorous ensemble interpretation* of quantum mechanics (QM) is useful to unveil precisely the actual core of EPR's discovery [29–31] which is the non-separability of the quantum world. This approach is rigorous insofar as it is *not* presupposed that each individual member of the ensemble always has (in the sense of EPR's principle of reality) precise values (properties) for all its property types. Instead, statements regarding properties to be ascribed to individual entities are completely avoided. Only the whole, i.e., the ensemble, will be considered as an element of QM, and QM is thought to make predictions about ensembles only. The being of the individual remains veiled.

Applying the rigorous ensemble approach to the teleportation problem, it is easy to show that the unknown state of an ensemble which is given by its statistical operator can be teleported successfully if the state in question and the state of the EPR ensemble are pure [32]. In contrast to the usual scheme, Alice has not to perform a measurement and transmit the result but she has to apply physical realizations of (mostly) projection operators and transmit *what* she has done. In accordance with the usual scheme, Alice destroys the unknown state by her activity. As soon as Bob obtains the

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information about what Alice has done, he is able to perform the necessary operations to impress the unknown (destroyed) state onto his subensemble. Remarkably, however, ensemble teleportation with fidelity 1 is possible even if Bob does not do anything with his subensemble at all! In order to achieve this automatic teleportation, Alice has to apply the physical realization of a special self-adjoint operator.

After an introduction into the ideal teleportation scheme for ensembles, we analyse the possibility and the fidelity of both conventional and automatic teleportation under more realistic conditions, i.e., we consider cases in which

- either one or both relevant states are mixed and
- the statistical operator of the EPR ensemble is not strictly non-separable but contaminated by a separable counterpart, respectively.

2. The ideal teleportation process

2.1. Preliminary remarks

Let the statistical operator ρ be defined on a 2^3-dimensional Hilbert space $\mathcal{H}_{\text{total}} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ where the set $\{|\alpha_1\rangle, |\alpha_2\rangle\}$ forms an orthonormal basis of the subspace $\mathcal{H}_A$, and let respective basis sets be given analogously for the other subspaces. In the four-dimensional subspace $\mathcal{H}_A \otimes \mathcal{H}_B$, we generate a Bell-type basis according to

$$|\Psi_e^{\pm}\rangle := \frac{1}{\sqrt{2}}\,(|\alpha_1\rangle|\beta_1\rangle \pm |\alpha_2\rangle|\beta_2\rangle), \tag{1}$$

$$|\Psi_o^{\pm}\rangle := \frac{1}{\sqrt{2}}\,(|\alpha_1\rangle|\beta_2\rangle \pm |\alpha_2\rangle|\beta_1\rangle). \tag{2}$$

With the aid of these Bell-basis, four statistical operators can be defined,

$$\begin{aligned}\rho_{1,2} &:= |\Psi_e^{\pm}\rangle\,\langle\Psi_e^{\pm}| \\ &= \tfrac{1}{2}\,(\hat{A}_{11}\otimes\hat{B}_{11} \pm \hat{A}_{12}\otimes\hat{B}_{12} \pm \hat{A}_{21}\otimes\hat{B}_{21} + \hat{A}_{22}\otimes\hat{B}_{22}),\end{aligned} \tag{3}$$

$$\begin{aligned}\rho_{3,4} &:= |\Psi_o^{\pm}\rangle\,\langle\Psi_o^{\pm}| \\ &= \tfrac{1}{2}\,(\hat{A}_{11}\otimes\hat{B}_{22} \pm \hat{A}_{12}\otimes\hat{B}_{21} \pm \hat{A}_{21}\otimes\hat{B}_{12} + \hat{A}_{22}\otimes\hat{B}_{11}),\end{aligned} \tag{4}$$

where $A_{ij} = |\alpha_i\rangle\langle\alpha_j|$ and $B_{ij} = |\beta_i\rangle\langle\beta_j|$. For these four operators, the following statements are valid:

- $\rho_i^2 = \rho_i$.
- $\rho_i\rho_j = 0 \quad \forall \quad i \neq j$.
- They are non-separable [30].

2.2. A teleportation scheme for ensembles

Suppose we are in possession of a generator producing an ensemble $\{(\mathrm{AB})_i\}$ of micro-entities $(\mathrm{AB})_i$ each of them dissociating according to $(\mathrm{AB})_i \longrightarrow \mathrm{A}_i + \mathrm{B}_i$. The subensemble of all A_i is sent either one by one or as a whole to Alice whereas the subensemble of all B_i is sent correspondingly to Bob. The ensemble $\{\{\mathrm{A}_i\}, \{\mathrm{B}_i\}\}$ consisting of the two subensembles is represented, say, by the statistical operator ρ_4 (see equation (4)) which means that, for the moment, we consider the ensemble in a pure state.

Now Alice obtains a further entity-ensemble, $\{\mathrm{C}_i\}$, which is assumed to be in a pure state as well. Then the new total ensemble $\{\{\mathrm{C}_i\}, \{\mathrm{A}_i\}, \{\mathrm{B}_i\}\}$ is represented by

$$\rho_{\text{total}} = \rho_C \otimes \rho_4, \tag{5}$$

where ρ_C is given by

$$\rho_C = \sum_{k,l=1}^{2} c_{kl}\hat{C}_{kl}. \tag{6}$$

The actual values of the coefficients c_{kl} must not be known to Alice.

$$\begin{aligned}&\Rightarrow \rho_{\text{total}} = \tfrac{1}{2}\,(c_{11}\hat{C}_{11} + c_{12}\hat{C}_{12} + c_{21}\hat{C}_{21} + c_{22}\hat{C}_{22}) \\ &\otimes\, (\hat{A}_{11}\otimes\hat{B}_{22} - \hat{A}_{12}\otimes\hat{B}_{21} - \hat{A}_{21}\otimes\hat{B}_{12} + \hat{A}_{22}\otimes\hat{B}_{11}).\end{aligned} \tag{7}$$

We define four new statistical operators, ρ_1', ρ_2', ρ_3' and ρ_4', so that they are analogous to the already introduced operators ρ_1, ρ_2, ρ_3 and ρ_4, respectively. The primed operators, however, shall act on $\mathcal{H}_C \otimes \mathcal{H}_A$ instead of $\mathcal{H}_A \otimes \mathcal{H}_B$. With the aid of these new operators, and after some lengthy but straightforward manipulations, ρ_{total} can be brought into the form

$$\begin{aligned}2\rho_{\text{total}} = \rho_1' \otimes (\sigma_3\sigma_1\rho_B\sigma_1\sigma_3) + \rho_2' \otimes (\sigma_1\rho_B\sigma_1) + \rho_3' \otimes (\sigma_3\rho_B\sigma_3) \\ + \rho_4' \otimes \rho_B + [\cdots],\end{aligned} \tag{8}$$

where we have made use of

$$\rho_B = c_{11}\hat{B}_{11} + c_{12}\hat{B}_{12} + c_{21}\hat{B}_{21} + c_{22}\hat{B}_{22}, \tag{9}$$

and employed the Pauli matrices σ_1 and σ_3, both related to the subspace $\mathcal{H}_B$. (The bulky last term on the rhs of (8) is shown in detail in [32]. By forming the partial trace (see below), it vanishes completely.)

Now, in the usual teleportation scheme formulated on the basis of wavefunctions of individual objects, Alice *measures* by which of the four states $|\Phi_e^{\pm}\rangle = 1/\sqrt{2}\,(|\gamma_1\rangle|\alpha_1\rangle \pm |\gamma_2\rangle|\alpha_2\rangle)$ and $|\Phi_o^{\pm}\rangle = 1/\sqrt{2}\,(|\gamma_1\rangle|\alpha_2\rangle \pm |\gamma_2\rangle|\alpha_1\rangle)$, respectively, her combined system C+A is represented. Recall, however, that this operation changes the state of the total system C+A+B too. After the measurement, Alice informs Bob by use of a classical transmission channel about the result so that Bob knows what to do in order to transform the state of his B into the original state of C. In this way, a state can be teleported, but in contrast to this scheme we will see below that in the case of ensembles things are different.

What if Alice *changes* the hitherto unknown state of C+A instead of measuring it? One could imagine that Alice applies one of four possible preparation tools to influence the state of C+A. Assume for the moment that Alice's operation on her subensemble C+A consists in the projection onto ρ_1', i.e., she acts upon the total system C+A+B by use of the operator $\rho_1' \otimes \hat{1}_B$. Then the 'expectation value' $\langle\rho_1' \otimes \hat{1}_B\rangle$ of her activity is the partial trace

$$\mathrm{Tr}_{\mathrm{C,A}}\left((\rho_1' \otimes \hat{1}_B)\rho_{\text{total}}\right). \tag{10}$$

Due to the mutual orthogonality of the primed operators, we obtain

$$\mathrm{Tr}_{\mathrm{C,A}}((\rho_1' \otimes \hat{1}_B)\rho_{\mathrm{total}}) = \tfrac{1}{4}(-c_{22}\hat{B}_{11} + c_{21}\hat{B}_{12} + c_{12}\hat{B}_{21} - c_{11}\hat{B}_{22}) + \tfrac{1}{2}\sigma_3\sigma_1\rho_B\sigma_1\sigma_3. \quad (11)$$

Insertion of ρ_B from equation (9) yields

$$\mathrm{Tr}_{\mathrm{C,A}}((\rho_1' \otimes \hat{1}_B)\rho_{\mathrm{total}}) = \tfrac{1}{4}\sigma_3\sigma_1\rho_B\sigma_1\sigma_3. \quad (12)$$

So Alice's 'expectation value' is an operator which refers to Bob's subensemble. It is seen immediately that this operator has a trace of $1/4$, i.e., it must be renormalized by division by its norm.

$$\Rightarrow \tilde{\mathrm{T}}\mathrm{r}_{\mathrm{C,A}}((\rho_1' \otimes \hat{1}_B)\rho_{\mathrm{total}}) \equiv \tilde{\rho}_{\mathrm{Bob}} = \sigma_3\sigma_1\rho_B\sigma_1\sigma_3. \quad (13)$$

Now this expression is a statistical operator representing the state of Bob's subensemble after the preparation of Alice. Recall that every influence exerted on A automatically changes B as well, because the subensembles A and B are in a non-separable state. So (13) describes what Bob has at hand. Subsequently Alice must tell Bob what she has done. It is important to note that she does *not* have to communicate any measurement result which would mean that she had to become aware of a pointer position before contacting Bob. After Alice has sent some bits of classical information to Bob, he will be able to apply some operations so that his subensemble B is represented by ρ_B which is the analogue of the unknown ρ_C in Bob's Hilbert space. In this way, teleportation of a statistical operator is achieved.

2.3. Automatic teleportation

In the most general case, Alice's preparation consists of the application of the operator

$$\hat{P} = \sum_{k,l,m,n} u_{klmn}|\gamma_k\rangle\langle\gamma_l| \otimes |\alpha_m\rangle\langle\alpha_n| \equiv \sum_{k,l,m,n} u_{klmn}\,\hat{C}_{kl} \otimes \hat{A}_{mn}. \quad (14)$$

Proceeding in the same way as described above, the state of Bob's subensemble will be transformed into

$$\tilde{\rho}_{\mathrm{Bob}} = \frac{\rho_{\mathrm{Bob}}}{\mathrm{Tr}(\rho_{\mathrm{Bob}})}, \quad (15)$$

where ρ_{Bob} can be written vectorially in the basis of the operators $\hat{B}_{ij}$ as follows:

$$\rho_{\mathrm{Bob}} = \frac{1}{2}\begin{pmatrix} c_{11}u_{1122} + c_{12}u_{2122} + c_{21}u_{1222} + c_{22}u_{2222} \\ -c_{11}u_{1112} - c_{12}u_{2112} - c_{21}u_{1212} - c_{22}u_{2212} \\ -c_{11}u_{1121} - c_{12}u_{2121} - c_{21}u_{1221} - c_{22}u_{2221} \\ c_{11}u_{1111} + c_{12}u_{2111} + c_{21}u_{1211} + c_{22}u_{2211} \end{pmatrix}. \quad (16)$$

This vector results from the original vector $c = (c_{11}, c_{12}, c_{21}, c_{22})$ (see equation (6)) by the transformation

$$\mathbf{T} = \begin{pmatrix} u_{1122} & u_{2122} & u_{1222} & u_{2222} \\ -u_{1112} & -u_{2112} & -u_{1212} & -u_{2212} \\ -u_{1121} & -u_{2121} & -u_{1221} & -u_{2221} \\ u_{1111} & u_{2111} & u_{1211} & u_{2211} \end{pmatrix}, \quad (17)$$

i.e.,

$$\tilde{\rho}_{\mathrm{Bob}} = \frac{(1/2)\mathbf{T}\vec{c}}{\|(1/2)\mathbf{T}\vec{c}\|}, \quad (18)$$

where we have presupposed that C is in a pure state which implies $\|\vec{c}\| = 1$. So Bob can impress the original state $\vec{c}$ on his subensemble by applying the inverse transformation T^{-1}. Here it is presupposed that the inverse actually exists which is always the case if Alice projects on one of the ρ_i'.

But what would happen if Bob completely renounced any manipulation of his $\tilde{\rho}_{\mathrm{Bob}}$, i.e., if he did not apply the inverse transformation? We start the investigation of this case by rewriting Bob's statistical operator in the form

$$\tilde{\rho}_{\mathrm{Bob}} = \vec{c} + \left(\frac{\mathbf{T}}{\|\mathbf{T}\vec{c}\|} - \mathbf{1}\right)\vec{c}, \quad (19)$$

where the second term represents the contamination of $\vec{c}$ due to the fact that Bob has not done anything at all. However, if we now choose

$$\mathbf{T} = \alpha\,\mathbf{1}, \quad (20)$$

with *any positive* α, we immediately obtain $\tilde{\rho}_{\mathrm{Bob}} = \vec{c}$ (in the basis of the $\hat{B}_{ij}$), if $\vec{c}$ is normalized, i.e., in this case Bob would *have* the teleported state of C after Alice's preparation without doing anything at all! This situation raises the following question: can the condition (20) imposed on T be realized by an operator $\hat{P}$? Then Alice would be able to prepare the total ensemble in a way that the *unknown* state $\vec{c}$ is teleported *automatically*.

It is easy to see that the necessary operator for automatic teleportation is given by

$$\hat{P}_{\mathrm{aut}} = \alpha(\hat{C}_{11} \otimes \hat{A}_{22} - \hat{C}_{21} \otimes \hat{A}_{12} - \hat{C}_{12} \otimes \hat{A}_{21} + \hat{C}_{22} \otimes \hat{A}_{11}). \quad (21)$$

The comparison of this equation and (4) shows that $P_{\mathrm{aut}} = \rho_4'$, if we choose $\alpha = 1/2$, i.e., automatic teleportation can be achieved by application of the well known ρ_4'.

It is important to note that automatic teleportation does not imply superluminal transmission of information. Alice changes the state of the *complete* system A+B+C which has to be considered as *one* entity. There is no signal travelling faster than light which could impress the unknown state of C onto Bob's B. Instead Alice manipulates the whole, and Bob's B is left over in the former state of C.

3. Deviations from the ideal conditions

3.1. Definition of the fidelity

We will consider three different types of perturbations which may appear if we pass from the ideal scheme discussed above to more realistic situations:

1. The state of C is not pure but a mixed one.
2. The state of the EPR ensemble A+B is not pure but a mixed one.
3. The EPR ensemble is not completely entangled, i.e., its state is represented by a convex sum of a non-separable and a separable operator.

The influence of these imperfections on the success of the teleportation is measured by use of the so-called fidelity f, which is defined according to

$$f = \mathrm{Tr}(\rho_C|_{\mathrm{Ensemble\ B}} \cdot \tilde{\rho}_{\mathrm{Bob}}). \quad (22)$$

However, this definition works well only as long as both states are pure. Assume that the input state $\rho_C|_{Ensemble\,B} \equiv \rho_{\text{in}}$ is mixed and *equal* to the output state $\tilde{\rho}_{\text{Bob}} \equiv \rho_{\text{out}} \equiv \rho$. In this case, we obtain $f = \text{Tr}\rho^2 < 1$, which is not a rational result. To circumvent this difficulty, several proposals for better fidelity definitions have been made. The most prominent of them is the Uhlmann fidelity [33] which is given by

$$f_U = \text{Tr}\left(\sqrt{\rho_{\text{out}}}\,\rho_{\text{in}}\,\sqrt{\rho_{\text{out}}}\right)^{1/2}, \tag{23}$$

but we will see that also f_U is not able to discriminate different teleportation outcomes sufficiently. Let, e.g., the state of the incoming ensemble be pure, i.e.,

$$\rho_{\text{in}} = \frac{1}{2}\begin{pmatrix} 1 & e^{i\varphi} \\ e^{-i\varphi} & 1 \end{pmatrix}. \tag{24}$$

If the teleportation process yields the maximally mixed counterpart of ρ_{in} which is $\rho_{\text{out}} = 1/2\,\mathbf{1}$, then the Uhlmann fidelity amounts to $1/\sqrt{2}$. If, however, a projector as

$$\rho_{\text{out}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{25}$$

results, then f_U adopts the *same* value despite the fact that the two teleportation outcomes are significantly different.

Also the trace distance of ρ_{in} and ρ_{out} does not allow for a good fidelity measure. With ρ_{in} given by (24) and $\rho_{\text{out}} = 1/2\,\mathbf{1}$, we arrive at $f = 1 - |\text{Tr}\rho_{\text{in}} - \text{Tr}\rho_{\text{out}}| = 1$, which does not make any sense. Moreover, if ρ_{out} is given by (25) instead of (24), nevertheless the same fidelity of 1 is obtained. These results demonstrate the necessity of an improved fidelity definition.

We propose the following measure of fidelity:

$$f_K = \left| 1 - \frac{\text{Tr}(\rho_{\text{in}} - \rho_{\text{out}})^2}{\text{Tr}\rho_{\text{in}}^2} \right|. \tag{26}$$

If input and output are identical, then $f_K = 1$ irrespective of the character of the state. Now let ρ_{in} be given by (24). With $\rho_{\text{out}} = 1/2\,1$, we obtain $f_K = 1/2$, i.e., the complete loss of the non-diagonal elements during teleportation manifests itself in our fidelity measure more pronouncedly than in Uhlmann's. If, however, the teleportation result is represented by ρ_{out} from (25), we end up with $f_K = 0$. So we do not only discriminate between the two outputs, but since in the latter case input and output have nothing in common, a fidelity of zero is in fact what one would expect.

Automatic teleportation implies that Bob does not do anything at all, i.e., the state of his subensemble is given by ρ_{Bob} at first. As long as Alice uses one of the four ρ_i', the trace of ρ_{Bob} is always equal to $1/4$. We therefore obtain

$$\tilde{\rho}_{\text{Bob}} = 4\,\rho_{\text{Bob}} \tag{27}$$

or, in terms of the vector representation,

$$\tilde{\rho}_{\text{Bob}} = 2\,\mathbf{T}\,\vec{c} \neq \vec{c}. \tag{28}$$

If, on the other hand, Bob performs the back transformation as usual, the new state of his subensemble is

$$\left(\tfrac{1}{2}\,\mathbf{T}^{-1}\right)\tilde{\rho}_{\text{Bob}} = \vec{c}, \tag{29}$$

i.e., he will always end up with the desired result so that $f_K = 1$. Obviously, the application of the correct back-transformation by Bob always guarantees a perfect teleportation. But what is the influence of a non-ideal scheme on automatic teleportation? This will be the topic of the following subsections.

3.2. The state of C is mixed

In this case, we replace (6) by

$$\rho_C^m = c_{11}\hat{C}_{11} + \gamma\,c_{12}\hat{C}_{12} + \gamma\,c_{21}\hat{C}_{21} + c_{22}\hat{C}_{22}, \tag{30}$$

where $0 \leqslant \gamma < 1$. $\gamma = 1$ is equivalent to using the pure operator ρ_C given by (6). The effect of the mixing parameter γ will be maximal if the non-diagonal coefficients are maximal as well. So we choose

$$c_{12} = r e^{i\phi} \quad \text{with} \quad r = \sqrt{c_{11} - c_{11}^2}, \tag{31}$$

where we have made use of $c_{22} = 1 - c_{11}$ and of the positivity of the spectrum of ρ_C. As we have already shown above, the subensemble B is transformed by Alice's activities into the renormalized state

$$\tilde{\rho}_{\text{Bob}} = 2\,\mathbf{T}\,\vec{c}, \tag{32}$$

where now, however, the transformation matrix is given by

$$\mathbf{T} = \begin{pmatrix} u_{1122} & u_{2122} & u_{1222} & u_{2222} \\ -u_{1112} & -u_{2112} & -u_{1212} & -u_{2212} \\ -u_{1121} & -u_{2121} & -u_{1221} & -u_{2221} \\ u_{1111} & u_{2111} & u_{1211} & u_{2211} \end{pmatrix}. \tag{33}$$

In order to make statements about the fidelity, the activity of Alice has to be fixed. In the following, we will assume that Alice projects by use of one of the four operators ρ_i', i.e., we obtain the following four transformation matrices:

$$\mathbf{T}_1 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

$$\mathbf{T}_2 = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}, \tag{35}$$

$$\mathbf{T}_3 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}, \tag{36}$$

$$\mathbf{T}_4 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. \tag{37}$$

What about automatic teleportation? If Alice chooses either ρ_1' or ρ_2', the fidelity becomes dependent on the C-phase ϕ,

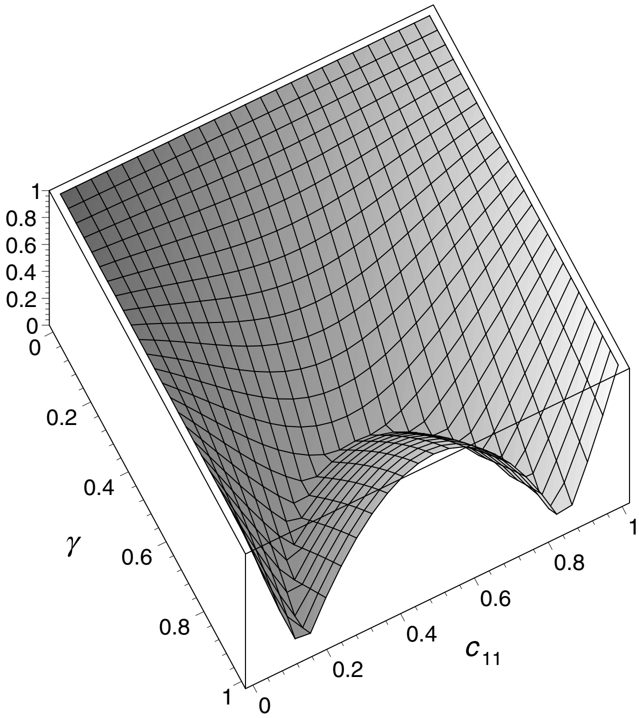

Figure 1. Fidelity $f_{K,3}(\gamma, c_{11})$, if the state of C is mixed subject to $1-\gamma$.

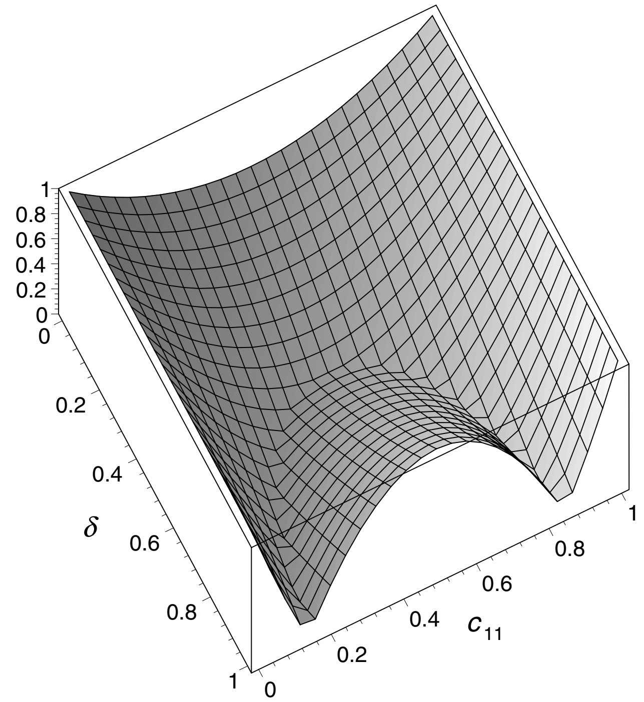

Figure 2. Fidelity $f_{K,3}(\delta, c_{11})$, if the state of A+B is mixed subject to $1-\delta$.

but since ϕ is unknown to both Alice and Bob, a reliable automatic teleportation is not possible at all. If, however, Alice chooses ρ_3', then the fidelity is phase-independent and given by

$$f_{K,3} = \left| 1 - \frac{8\,\gamma^2\,(c_{11} - c_{11}^2)}{1 - 2\,(1-\gamma^2)\,(c_{11} - c_{11}^2)} \right| . \tag{38}$$

The result of the numerical evaluation of $f_{K,3}$ can be found in figure 1. In general, the teleportation success is poor, but at least for all *maximally* mixed input states perfect teleportation can be achieved. Employing ρ_4' instead, it is seen immediately that $\tilde{\rho}_{\mathrm{Bob}} = \rho_C^m$ so that $f_{K,4} = 1$ for all possible income states.

3.3. The state of A+B is mixed

Now the operator ρ_4 in (5) has to be replaced by

$$\rho_4^m = \tfrac{1}{2}(\hat{A}_{11} \otimes \hat{B}_{22} - \delta\,\hat{A}_{12} \otimes \hat{B}_{21} - \delta\,\hat{A}_{21} \otimes \hat{B}_{12} + \hat{A}_{22} \otimes \hat{B}_{11}), \tag{39}$$

where the mixing parameter δ is bounded by $0 \leqslant \delta < 1$. In contrast to the previous case, this parameter enters the general formula of the transformation matrix $\mathbf{T}$:

$$\mathbf{T} = \begin{pmatrix} u_{1122} & u_{2122} & u_{1222} & u_{2222} \\ -\delta\,u_{1112} & -\delta\,u_{2112} & -\delta\,u_{1212} & -\delta\,u_{2212} \\ -\delta\,u_{1121} & -\delta\,u_{2121} & -\delta\,u_{1221} & -\delta\,u_{2221} \\ u_{1111} & u_{2111} & u_{1211} & u_{2211} \end{pmatrix} . \tag{40}$$

Assume again that Alice projects by use of one of the four operators ρ_i'. The corresponding matrices are equal to the $\mathbf{T}_i$ given by (34)–(37) with the exception that now the elements of the second and the third row must be multiplied by δ.

For automatic teleportation, we arrive at the same problem as discussed above, if Alice employs either ρ_1' or ρ_2', i.e., the fidelity becomes phase-dependent and in consequence the mission is bootless. If, however, Alice applies one of the other projectors, we end up with

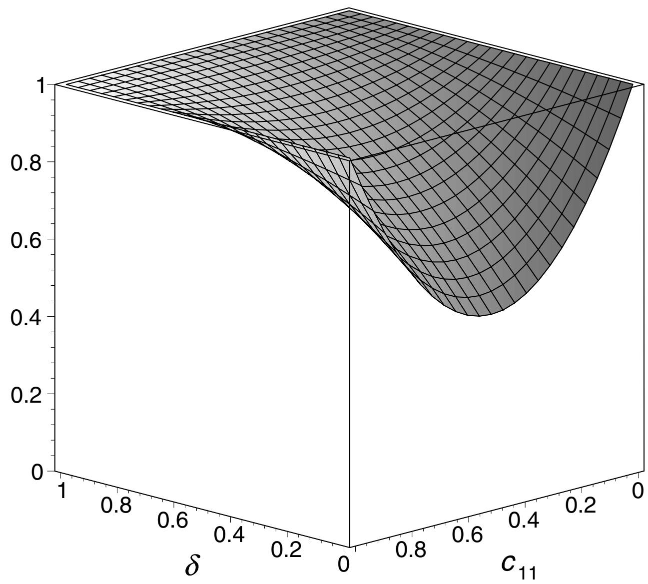

Figure 3. Fidelity $f_{K,4}(\delta, c_{11})$, if the state of A+B is mixed subject to $1-\delta$.

$$f_{K,3,4} = \left| 1 - 2\,(1 \pm \delta)^2\,(c_{11} - c_{11}^2) \right| . \tag{41}$$

The results of the numerical evaluation can be found in figures 2 and 3, respectively. Whereas with ρ_3' perfect teleportation is possible only in trivial cases, ρ_4' allows for a fidelity larger than 0.95 as long as $1-\delta$, the degree of mixedness, amounts to less than 0.3162.

It is worth to mention that the relation between mixedness and teleportation has been investigated some years ago by

Bose and Vedral [34] who have quantified the mixedness by the von Neumann entropy. They found that there is a certain entropy threshold which, if exceeded, makes A+B useless for teleportation. Here we have characterized mixedness by $1-\delta$, and it is seen immediately that $f_{K,4}$ exceeds the classical 2/3 limit if and only if $1-\delta < 1-\sqrt{2/3} = 0.8165$.

3.4. The EPR ensemble is noisy

In this case, we assume that the source generating A+B does not operate perfectly, which means that the ensemble A+B is not completely entangled so that it has to be represented by a superposition of a non-separable and a separable statistical operator. We therefore replace ρ_4 by

$$\rho_4^{ps} = \varepsilon\rho_4 + (1-\varepsilon)\rho_{\text{sep}}, \tag{42}$$

where for the separable part the ansatz

$$\rho_{\text{sep}} = \rho_A \otimes \rho_B = \sum_{i,j,k,l} (a_{ij}\hat{A}_{ij}) \otimes (b_{kl}\hat{B}_{kl}) \tag{43}$$

is made. Note that this definition of separability deviates from the usual one. The reasons for employing (43) are discussed in detail in [35]. Let us assume for the sake of simplicity that Alice projects onto ρ_4'. Then, according to (10),

$$\rho_{\text{Bob}} = \text{Tr}_{\text{C,A}}\left((\rho_4' \otimes \hat{1}_B)(\rho_C \otimes \rho_4^{ps})\right). \tag{44}$$

A lengthy but straightforward calculation yields

$$\rho_{\text{Bob}} = \left(\frac{\varepsilon}{4}c_{11} + \frac{1-\varepsilon}{2}z\,b_{11}\right)\hat{B}_{11} + \left(\frac{\varepsilon}{4}c_{12} + \frac{1-\varepsilon}{2}z\,b_{12}\right)\hat{B}_{12} + \left(\frac{\varepsilon}{4}c_{21} + \frac{1-\varepsilon}{2}z\,b_{21}\right)\hat{B}_{21} + \left(\frac{\varepsilon}{4}c_{22} + \frac{1-\varepsilon}{2}z\,b_{22}\right)\hat{B}_{22}, \tag{45}$$

with

$$z = c_{11}a_{22} - c_{12}a_{21} - c_{21}a_{12} + c_{22}a_{11}. \tag{46}$$

We will now investigate two special cases: let first of all $a_{ij} = b_{kl} = 1/2\,\forall\, i, j, k, l$. In this case, both ρ_A and ρ_B are pure states. With (31) we obtain the trace over ρ_{Bob} as

$$\text{Tr}\rho_{\text{Bob}} = \tfrac{1}{4}\underbrace{[1 - 2(1-\varepsilon)c_{11}(1-c_{11})\cos\phi]}_{:=\alpha}, \tag{47}$$

so that

$$\tilde{\rho}_{\text{Bob}} = \frac{4}{\alpha}\rho_{\text{Bob}} = \frac{2}{\alpha}\mathbf{T}\vec{c}, \tag{48}$$

with

$$\mathbf{T} = \frac{1}{4}\begin{pmatrix} \varepsilon+1 & \varepsilon-1 & \varepsilon-1 & 1-\varepsilon \\ 1-\varepsilon & 3\varepsilon-1 & \varepsilon-1 & 1-\varepsilon \\ 1-\varepsilon & \varepsilon-1 & 3\varepsilon-1 & 1-\varepsilon \\ 1-\varepsilon & \varepsilon-1 & \varepsilon-1 & \varepsilon+1 \end{pmatrix}. \tag{49}$$

The final result is a very complicated function of the entanglement degree ε, of c_{11}, and of the phase ϕ contained in $\vec{c}$. We wil therefore evaluate f_K for some special cases only. Let first ϕ be equal to 0. The fidelity is then given by

$$\beta := \left(4\sqrt{c_{11}(1-c_{11})} + 8c_{11}^4 - 16c_{11}^3 + 14c_{11}^2 - 6c_{11}\right), \tag{50}$$

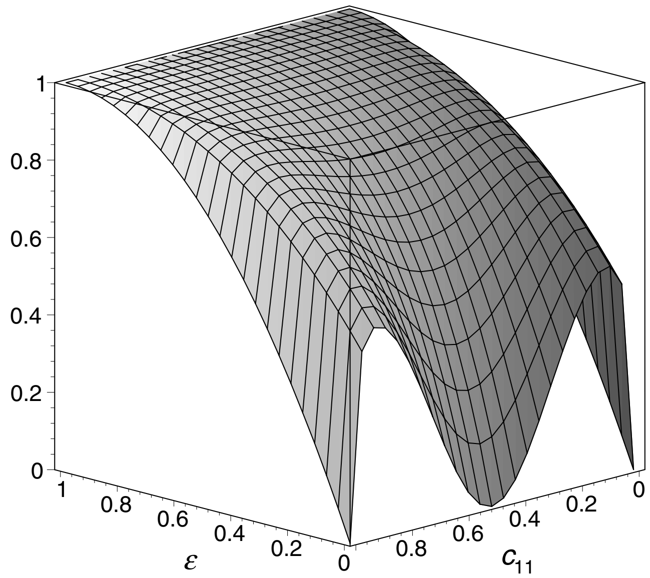

Figure 4. Fidelity $f_K(\varepsilon, c_{11})$, if A+B is noisy subject to $1-\varepsilon$ and $\phi = 0$.

$$f_K = \frac{\beta\,(\varepsilon-1)^2 - 4(c_{11}^2 - c_{11})(\varepsilon-1) - \varepsilon^2 + 2\varepsilon}{(2\varepsilon c_{11}^2 - 2\varepsilon c_{11} - 2c_{11}^2 + 2c_{11} - 1)^2}. \tag{51}$$

The characteristics of the fidelity is shown in figure 4 as a function of ε and c_{11}. In the case of a completely destroyed EPR ensemble ($\varepsilon = 0$) we obtain a fidelity of 0 for all possible input states whereas in the opposite case $f_K = 1$, as expected.

If, on the other hand, $\phi = \pi/2$, (51) is simplified significantly to

$$f_K = 2\varepsilon - \varepsilon^2, \tag{52}$$

i.e., now the fidelity is dependent on the entanglement degree only and varies from 0 to 1.

Further analysis shows that for a contamination degree $(1-\varepsilon)$ of 0.1 a fidelity of *at least* 0.98 can be obtained irrespective of the phase ϕ. Assuming a 50:50 mixture of entangled and disentangled systems ($\varepsilon = 1/2$), the fidelity still has a *lower* limit of 8/9 as long as the nearly trivial cases with c_{11} in the vicinity of 0 or 1 are excluded.

With the conditions $a_{ii} = b_{kk} = 1/2\,\forall\, i, k$ and $a_{ij} = b_{kl} = 0$ otherwise, we produce the maximally mixed counterpart to the first case. In contrast to (51), the fidelity is now a very simple function of ε only:

$$f_K = 1 - \frac{(1-\varepsilon)^2}{2}. \tag{53}$$

For $\varepsilon = 1$ perfect agreement between the input and the output state is obtained, and for $\varepsilon = 0$ we arrive at $f_K = 1/2$. In this case, ρ_{out} is the maximally mixed counterpart of ρ_{in}, i.e., the fidelity matches the result which has been derived in subsection 3.1.

4. Summary

Under certain circumstances, automatic teleportation is possible, i.e., the unknown state of an ensemble C can be conferred to a subensemble B which is EPR-correlated with its counterpart A, if the physicist Alice who is in possession of C and A performs a certain projection. In the present contribution, we have analysed the possibility of automatic teleportation under suboptimal conditions, using an improved fidelity measure, and obtained the following results:

1. Assume that C is not in a pure but in a mixed state, where the degree of mixedness is described by a parameter $1-\gamma$ with $0 \leqslant \gamma < 1$ so that $1-\gamma = 1$ indicates maximal mixedness. If Alice makes use of ρ_4', the input state is teleported automatically with fidelity $= 1$ irrespective of the value of $1-\gamma$. If, however, Alice employs ρ_3', the success of the automatic teleportation depends strongly on the degree of mixedness. For $1-\gamma > 0.6985$, a fidelity larger than $2/3$ can be achieved. This is easy to understand, because the smaller the non-diagonal parts of the input state are, so much the less is the state's information content, and so much better the automatic teleportation will work. Recall that the value of 2/3 is the upper bound of the fidelity if only classically correlated systems are used. Note that a *pure* ρ_C in combination with ρ_3' in general does not enable automatic teleportation.
2. The success of automatic teleportation will be affected, if the EPR ensemble A+B is not in a pure but in a mixed state. The corresponding parameter $1-\delta$ must be lower than 0.8165 to guarantee a fidelity larger than $2/3$, i.e., compared to the previous case the mixedness of the EPR ensemble is less important for the possibility to detect automatic teleportation.
3. The situation that the EPR ensemble is noisy (not optimally entangled) can be represented by adding a separable statistical operator to the non-separable one, where a coefficient $1-\varepsilon$ with $0 \leqslant \varepsilon < 1$ describes the weight of the separable part. If the separable statistical operator is pure, the fidelity becomes a complicated function also of the phase ϕ of the state of the input ensemble C. For $\phi = 0$ and $1-\varepsilon \leqslant 0.3654$, automatic teleportation with a fidelity of more than 0.95 is still possible, and the classical limit will be exceeded even if $1-\varepsilon = 0.7321$, i.e., a non-classical effect could be detected even if only 28% of the EPR ensemble is entangled. If, on the contrary, the separable operator is maximally mixed, the fidelity depends on ε only, and we obtain the soft detectability criterion $1-\varepsilon > \sqrt{2/3} = 0.8165$.

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